

7402 (2009).

Q1 (a). Put  $u = R(r)\Theta(\theta)$  and substitute:

$$\frac{1}{r} \frac{\partial}{\partial r} (rR'\Theta) + \frac{1}{r^2} R\Theta'' = 0$$

$$R''\Theta + \frac{1}{r} R'\Theta + \frac{1}{r^2} R\Theta'' = 0$$

$$\frac{r^2 R'' + rR'}{R} + \frac{\Theta''}{\Theta} = 0 \quad \text{so} \quad \frac{r^2 R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \mu.$$

Look at the  $\Theta$ -equation first:

$$\Theta'' + \mu\Theta = 0; \quad \Theta = A\sin(\sqrt{\mu}\theta) + B\cos(\sqrt{\mu}\theta).$$

$$\text{Then } u(r, 0) = 0 \Rightarrow \Theta(0) = 0 \Rightarrow B = 0$$

$$u(r, \pi) - \frac{\partial u}{\partial \theta}(r, \pi) = 0 \Rightarrow \Theta(\pi) - \Theta'(\pi) = 0 \Rightarrow \sin(\sqrt{\mu}\pi) - \sqrt{\mu}\cos(\sqrt{\mu}\pi)$$

This is only possible if  $\sqrt{\mu}$  is real, so put  $\mu = \lambda^2$ ,  $\lambda > 0$ .

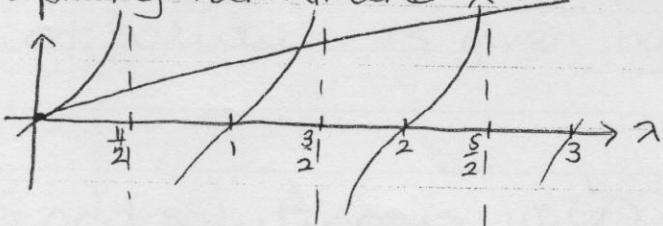
(Note: if we try  $\mu = 0$ ,  $\Theta = A\theta + B$ ;  $B = 0$ ;  $A = 0$ : no good.)

$$\text{Thus } \Theta'' + \lambda^2\Theta = 0.$$

$$r^2 R'' + rR' - \lambda^2 R = 0.$$

$$\text{We need } \sin(\lambda\pi) - \lambda\cos(\lambda\pi) = 0 \Rightarrow \tan(\lambda_n\pi) = \lambda_n.$$

(b). Graphing  $\tan \lambda\pi$  and  $\lambda$ :



We can see there is one root in each integer range  $[n: n + \frac{1}{2}]$ .

(c). The  $R$  equation is solved by  $r^\alpha$ , where

$$\alpha(\alpha-1) + \alpha - \lambda^2 = 0 \quad \alpha^2 = \lambda^2 \quad \alpha = \pm \lambda.$$

We need  $R \rightarrow 0$  as  $r \rightarrow \infty$  so use only  $r^{-\lambda}$ :

$$u(r, \theta) = \sum_{n=1}^{\infty} A_n \sin \lambda_n \theta r^{-\lambda_n}.$$

$$u(a, \theta) = f(\theta) = \sum_{n=1}^{\infty} A_n \sin \lambda_n \theta a^{-\lambda_n}$$

$\infty$  by orthogonality,

$$A_n a^{-\lambda_n} = \int_0^\pi \frac{\sin \lambda_n \phi f(\phi)}{\int_0^\pi \sin^2 \lambda_n \phi d\phi} d\phi$$

$$\frac{\partial u}{\partial r}(a, \theta) = \sum_{n=1}^{\infty} (-\lambda_n) a^{-\lambda_n-1} A_n \sin \lambda_n \theta$$

$$= \sum_{n=1}^{\infty} -\frac{\lambda_n}{a} \frac{\int_0^\pi \sin \lambda_n \phi f(\phi) d\phi}{\int_0^\pi \sin^2 \lambda_n \phi d\phi} \sin \lambda_n \theta.$$

x2. We can write it as  $y'' + \frac{3}{x}y' + \frac{(1+x)}{x^2}y = 0$  so in the usual notation,  $p(x) = \frac{3}{x}$  and  $q(x) = \frac{1+x}{x^2}$  which are both singular; but  $xp(x) = 3$  and  $x^2q(x) = 1+x$  which are both analytic, so  $x=0$  is a regular singular point.

If we try  $y = \sum_{k=0}^{\infty} a_k x^{k+c}$  then we have

$$\sum_{k=0}^{\infty} a_k [(k+c)(k+c-1) + 3(k+c) + 1] x^{k+c} + \sum_{k=0}^{\infty} a_k x^{k+c+1} = 0$$

$$\sum_{k=0}^{\infty} a_k [(k+c)^2 + 2(k+c) + 1] x^{k+c} + \sum_{k=1}^{\infty} a_{k-1} x^{k+c} = 0.$$

Setting the coefficient = 0 for  $k \geq 1$  gives

$$a_k [(k+c+1)^2] + a_{k-1} = 0 \quad a_k = -\frac{a_{k-1}}{(k+c+1)^2}$$

The  $k=0$  condition gives  $(c+1)^2 = 0$ ,  $c = -1$ .

The first solution is given by  $a_k = \frac{(-1)^k [(c+1)!]^2}{[(k+c+1)!]^2}$ .

The second solution is  $w_2(x) = \frac{d}{dc} (w_1(x))$  by Frobenius evaluated at  $c = -1$

$$y = \frac{d}{dc} \sum_{k=0}^{\infty} \frac{(-1)^k [(c+1)!]^2}{[(k+c+1)!]^2} x^{k+c}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k [(c+1)!]^2}{[(k+c+1)!]^2} \ln x \cdot x^{k+c} + \sum_{k=1}^{\infty} (-1)^k x^{k+c} \frac{d}{dc} \left( \frac{[(c+1)!]^2}{[(k+c+1)!]^2} \right)$$

$$\text{Now } \frac{d}{dc} \left( \frac{(c+1)!}{(k+c+1)!} \right) = \frac{d}{dc} \left( \frac{1}{(k+c+1)(k+c) \dots (c+2)} \right) = \frac{-(c+1)!}{(k+c+1)!} \times \sum_{r=1}^k \frac{1}{(c+1+r)}$$

$$\Rightarrow y = \sum_{k=0}^{\infty} \frac{(-1)^k [0!]^2}{[k!]^2} \ln x \cdot x^{k-1} + \sum_{k=1}^{\infty} \frac{(-1)^k x^{k-1} \cdot 2(c+1)! \cdot (-1) \cdot (c+1)!}{(k+c+1)! (k+c+1)!} \underbrace{\sum_{r=1}^k \frac{1}{(c+1+r)}}_{S_k}$$

$$= \sum_{k=0}^{\infty} \frac{\ln x (-1)^k x^{k-1}}{(k!)^2} - \sum_{k=1}^{\infty} \frac{2(-1)^k S_k x^{k-1}}{(k!)^2}$$

Q3. (a).  $(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0 \quad [1]$   
 $(1-x^2)P_m'' - 2xP_m' + m(m+1)P_m = 0. \quad [2]$

Take  $P_m \times [1] - P_n \times [2]$  and integrate:

$$\int_{-1}^1 (1-x^2)P_n''P_m - (1-x^2)P_nP_m'' - 2xP_n'P_m + 2xP_nP_m' dx \\ + [n(n+1) - m(m+1)] \int_{-1}^1 P_nP_m dx = 0.$$

Integrate by parts:

$$[(1-x^2)P_n'P_m - (1-x^2)P_nP_m']_{-1}^1 - \int_{-1}^1 (-2x)P_n'P_m + (1-x^2)P_n'P_m' dx \\ + \int_{-1}^1 (-2x)P_nP_m' + (1-x^2)P_n'P_m dx + \int_{-1}^1 (-2x)P_n'P_m + 2xP_nP_m' dx \\ + [n(n+1) - m(m+1)] \int_{-1}^1 P_nP_m dx = 0.$$

Cancelling and noting  $(1-x^2)=0$  at  $x=\pm 1$ , we have

$$\{n(n+1) - m(m+1)\} \int_{-1}^1 P_nP_m dx = 0$$

so the integral is zero for  $n \neq m$ .

(b)  $\exp\left[\frac{1}{2}x(t-\frac{1}{t})\right] = \sum_{n=-\infty}^{\infty} t^n J_n(x).$

(i) Differentiate w.r.t.  $x$ :

$$\sum_{n=-\infty}^{\infty} t^n J_n'(x) = \frac{1}{2}\left(t - \frac{1}{t}\right) \exp\left[\frac{1}{2}x(t-\frac{1}{t})\right] \\ = \frac{1}{2} \sum_{n=-\infty}^{\infty} t^{n+1} J_n(x) - \frac{1}{2} \sum_{n=-\infty}^{\infty} t^{n-1} J_n(x) \\ = \frac{1}{2} \sum_{n=-\infty}^{\infty} t^n J_{n-1}(x) - \frac{1}{2} \sum_{n=-\infty}^{\infty} t^n J_{n+1}(x)$$

so  $J_n'(x) = \frac{1}{2}J_{n-1}(x) - \frac{1}{2}J_{n+1}(x).$

Q3 (b) (ii) Differentiate w.r.t.  $t$ :

$$\begin{aligned} \sum_{n=-\infty}^{\infty} nt^{n-1} J_n(x) &= \left(\frac{1}{2}x + \frac{1}{2}x \frac{1}{t^2}\right) \exp\left[\frac{1}{2}x(t - \frac{1}{t})\right] \\ &= \frac{1}{2}x \sum_{n=-\infty}^{\infty} t^n J_n(x) + \frac{1}{2}x \sum_{n=-\infty}^{\infty} t^{n-2} J_n(x) \\ &= \frac{1}{2}x \sum_{n=-\infty}^{\infty} t^{n-1} J_{n-1}(x) + \frac{1}{2}x \sum_{n=-\infty}^{\infty} t^{n-1} J_{n+1}(x) \end{aligned}$$

$$\text{so } n J_n(x) = \frac{1}{2}x \{J_{n-1}(x) + J_{n+1}(x)\}.$$

(iii) Put  $t = e^{i\theta}$ .

$$\begin{aligned} \sum_{n=-\infty}^{\infty} e^{inx} J_n(x) &= \exp\left(\frac{1}{2}x(e^{i\theta} - e^{-i\theta})\right) \\ &= \exp(ix \sin \theta) \\ &= \cos(x \sin \theta) + i \sin(x \sin \theta). \end{aligned}$$

Real parts:

$$\sum_{n=-\infty}^{\infty} \cos(n\theta) J_n(x) = \cos(x \sin \theta)$$

Integrate:

$$\begin{aligned} \int_0^{2\pi} \cos(x \sin \theta) d\theta &= \sum_{n=-\infty}^{\infty} J_n(x) \underbrace{\int_0^{2\pi} \cos(n\theta) d\theta}_{0 \text{ if } n \neq 0} \\ &= J_0(x) \cdot 2\pi \text{ as required.} \end{aligned}$$

$$\begin{aligned}
 Q4 (a). (\widehat{f * g}) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f * g)(x) e^{-ikx} dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y) g(y) e^{-iky} e^{-ik(x-y)} dy dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) e^{-iky} \int_{-\infty}^{\infty} f(x-y) e^{-ik(x-y)} dx dy
 \end{aligned}$$

Change from  $x$  to  $u = x - y$  (at fixed  $y$ ):

$$\begin{aligned}
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) e^{-iky} \int_{-\infty}^{\infty} f(u) e^{-iku} du dy \\
 &= \frac{1}{\sqrt{2\pi}} \left( \sqrt{2\pi} \hat{g}(k) \right) \left( \sqrt{2\pi} \hat{f}(k) \right) = \sqrt{2\pi} \hat{f}(k) \hat{g}(k).
 \end{aligned}$$

(b). Fourier transform in  $x$ :

$$\begin{aligned}
 \hat{u}(k, y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, y) e^{-ikx} dx \\
 \text{and } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{-ikx} dx &= \frac{1}{\sqrt{2\pi}} \left[ \frac{\partial u}{\partial x} e^{-ikx} \right]_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-ik) \frac{\partial u}{\partial x} e^{-ikx} dx \\
 &= \frac{ik}{\sqrt{2\pi}} \left\{ \left[ ue^{-ikx} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (-ik) u e^{-ikx} dx \right\} = -k^2 \hat{u}.
 \end{aligned}$$

$$\frac{\partial^2 \hat{u}}{\partial y^2} - k^2 \hat{u} = 0 ; \quad \hat{u} \rightarrow 0 \text{ as } y \rightarrow \infty$$

$$\hat{u}(k, 0) = \hat{f}(k).$$

The solution decaying in the farfield is  
 $\hat{u}(k, y) = A(k) \exp[-|k|y]$   
and the  $y=0$  BC gives  $A(k) = \hat{f}(k)$ .

$$\hat{u}(k, y) = \hat{f}(k) \exp[-|k|y].$$

Inverting,

$$u(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{-iky} e^{ikx} dk$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^0 \hat{f}(k) e^{ky} e^{ikx} dk + \int_0^{\infty} \hat{f}(k) e^{-ky} e^{ikx} dk \right\}$$

Substitute the definition of  $\hat{f}(k)$ :

$$u(x, y) = \frac{1}{\sqrt{2\pi}} \left\{ \int_{k=-\infty}^0 e^{ky} e^{ikx} \cdot \frac{1}{\sqrt{2\pi}} \int_{s=-\infty}^{\infty} f(s) e^{-iks} ds dk + \int_{k=0}^{\infty} e^{-ky} e^{ikx} \cdot \frac{1}{\sqrt{2\pi}} \int_{s=-\infty}^{\infty} f(s) e^{-iks} ds dk \right\}$$

$$= \frac{1}{2\pi} \left\{ \int_{s=-\infty}^{\infty} f(s) \left[ \int_{k=-\infty}^0 e^{k(y+ix-is)} dk + \int_{k=0}^{\infty} e^{k(ix-is-y)} dk \right] ds \right\}$$

$$= \frac{1}{2\pi} \int_{s=-\infty}^{\infty} f(s) \left\{ \left[ \frac{e^{k(y+ix-is)}}{(y+ix-is)} \right]_{-\infty}^0 + \left[ \frac{e^{k(ix-is-y)}}{(ix-is-y)} \right]_0^{\infty} \right\} ds$$

$$= \frac{1}{2\pi} \int_{s=-\infty}^{\infty} f(s) \left\{ \frac{1}{y+ix-is} - \frac{1}{ix-is-y} \right\} ds$$

$$= \frac{1}{2\pi} \int_{s=-\infty}^{\infty} f(s) \left\{ \frac{y-ix+is+y+ix-is}{(x-s)^2+y^2} \right\} ds$$

$$= \frac{y}{\pi} \int_{s=-\infty}^{\infty} \frac{f(s)}{(x-s)^2+y^2} ds$$

$$Q5. (a) \quad L[f] = \int_0^\infty e^{-st} f(t) dt.$$

$$(i) \quad L[e^{-at}] = \int_0^\infty e^{-st} e^{-at} dt = \int_0^\infty e^{-(s+a)t} dt = \left[ \frac{e^{-(s+a)t}}{-(s+a)} \right]_0^\infty \\ = -\left( \frac{1}{-(s+a)} \right) = \frac{1}{(s+a)}.$$

$$(ii) \quad L[1] = L[e^{ot}] = \frac{1}{(s+o)} = \frac{1}{s}.$$

$$(iii) \quad L\left[\frac{df}{dt}\right] = \int_0^\infty e^{-st} \frac{df}{dt} dt = [e^{-st} f]_0^\infty - \int_0^\infty (-s)e^{-st} f dt \\ = 0 - f(0) + s \int_0^\infty e^{-st} f dt = s \bar{f}(s) - f(0).$$

$$(b) \quad \frac{\partial u}{\partial t} + x \frac{\partial u}{\partial x} = x. \quad \text{L.T. in } t:$$

$$s\bar{u} - u(x, 0) + x \frac{\partial \bar{u}}{\partial x} = \frac{x}{s} \quad \text{with } \bar{u}(0, s) = \frac{1}{s}.$$

$$x\bar{u}_x + s\bar{u} = \frac{x}{s} + 1 + x^2. \quad \text{CF: } x^\alpha \Rightarrow \alpha + s = 0, \quad x^{-s}.$$

PI: Try  $Ax + B + Cx^2$ .

$$Ax + 2Cx^2 + sA\alpha + sB + sC\alpha^2 = \frac{x}{s} + 1 + x^2.$$

$$x//: A(1+s) = \frac{1}{s} \Rightarrow A = \frac{1}{s(1+s)}$$

$$1//: sB = 1 \Rightarrow B = \frac{1}{s}$$

$$x^2//: (2+s)C = 1 \Rightarrow C = \frac{1}{(s+2)}$$

$$\text{So } \bar{u} = \frac{x}{s(1+s)} + \frac{1}{s} + \frac{x^2}{(s+2)} + Dx^{-s}.$$

$$\bar{u}(0, s) = \frac{1}{s} + D \lim_{x \rightarrow 0} x^{-s} \text{ needs to be } \frac{1}{s} \Rightarrow D = 0.$$

$$\bar{u}(x, s) = \frac{x^2}{(s+2)} + \frac{x}{s(s+1)} + \frac{1}{s} = \frac{x^2}{(s+2)} + \frac{x}{s} - \frac{x}{(s+1)} + \frac{1}{s}$$

$$\Rightarrow u(x, t) = x^2 e^{-2t} + x - xe^{-t} + 1.$$

$$Q6. \quad u(x) + (f * u)(x) = g(x)$$

$$\hat{u}(k) + \sqrt{2\pi} \hat{f}(k) \hat{u}(k) = \hat{g}(k)$$

$$\Rightarrow \hat{u}(k) = \frac{\hat{g}(k)}{(1 + \sqrt{2\pi} \hat{f}(k))}$$

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{g}(k) e^{ikx}}{1 + \sqrt{2\pi} \hat{f}(k)} dk.$$

$$f(x) = \frac{1}{2} e^{-|x|} \Rightarrow \hat{f}(k) = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} e^{-ikx} dx = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} \cos kx dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x} \cos kx dx = \operatorname{Re} \left[ \int_{\sqrt{2\pi}}^{\infty} e^{-x} e^{ikx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \operatorname{Re} \left[ \frac{e^{x(-1+ik)}}{(-1+ik)} \right]_0^{\infty} = \frac{1}{\sqrt{2\pi}} \operatorname{Re} \left[ \frac{-1}{(-1+ik)} \right]$$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}(1+k^2)}$$

$$g(x) = e^{-x^2/2} \Rightarrow \hat{g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \cos kx dx$$

$$= \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi} e^{-k^2/2}.$$

$$\hat{g}(k) = e^{-k^2/2}.$$

$$\text{Then } u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-k^2/2} e^{ikx}}{(1 + \frac{1}{2}(1+k^2))} dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{(1+k^2)e^{-k^2/2}}{(2+k^2)} e^{ikx} dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-k^2/2} e^{ikx} dk - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-k^2/2}}{(2+k^2)} e^{ikx} dk$$

$$= e^{-x^2/2} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{(2+k^2)} e^{-k^2/2} e^{ikx} dk.$$

$$\text{Now } \int_{-\infty}^{\infty} \frac{e^{ikx}}{2+k^2} dk = \int_{-\infty}^{\infty} \frac{\cos kx}{2+k^2} dk = \frac{\pi}{\sqrt{2}} e^{-\sqrt{2}|x|} \text{ so if}$$

we define  $h(x) = \frac{\sqrt{\pi}}{2} e^{-\sqrt{2}|x|}$  then  $\hat{h}(k) = \frac{1}{2+k^2}$ .

Equally,  $e^{-x^2/2} = \hat{g}(k)$  so

$$u(x) = e^{-x^2/2} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{h}(k) \hat{g}(k) e^{ikx} dk$$

$$= e^{-x^2/2} - \frac{1}{2\pi} \int_{-\infty}^{\infty} (\hat{h} * \hat{g})(k) e^{ikx} dk$$

$$= e^{-x^2/2} - \frac{1}{\sqrt{2\pi}} (h * g)(x)$$

$$= e^{-x^2/2} - \frac{1}{\sqrt{2\pi}} \cdot \frac{\sqrt{\pi}}{2} \int_{s=-\infty}^{\infty} e^{-s^2/2} e^{-\sqrt{2}|x-s|} ds$$

$$= e^{-x^2/2} - \frac{1}{2\sqrt{2}} \int_{s=-\infty}^{\infty} e^{-s^2/2} e^{-\sqrt{2}|x-s|} ds.$$